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Finite-dimensional indecomposable representations of the generalised Lie algebras

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Abstract. We investigate a possible way of constructing finite-dimensional indecomposable representations of the generalised Lie algebras. The spl(2, 1) superalgebra has been studied as an example.

1. Introduction

Considering the importance of graded Lie algebras (GLA) in theoretical physics [1], especially semisimple GLA, it is of interest to examine such algebras and their representations in the search for a super unified group. The spl(2, 1) superalgebra is one such algebra whose even part is the Weinberg-Salam algebra and, in this context, it has received attention in several attempts in obtaining a super unified group of particle interactions [2].

In Lie algebra theory, the three statements stating the non-existence of an Abelian ideal, non-degeneracy of the Killing form and reducibility of all finite-dimensional representations are completely equivalent. In GLA theory, however, each statement is stronger than the other [3], and several semisimple superalgebras with a desired even part possess indecomposable representations [4-6], a feature which is naturally undesirable.

Infinite-dimensional indecomposable representations have been considered earlier by Gruber *et al* [4] for the osp(1, 2) superalgebra. In this paper, we attempt a construction of finite-dimensional indecomposable representations for arbitrary Γ graded Lie algebras [7, 8] (ΓGLA), which to the best of our knowledge has not been presented previously.

In an earlier communication [5], we had found a necessary and sufficient condition for finite-dimensional indecomposable representations using cohomology theory, namely, that the first cohomology group of the algebra g on a set of endomorphisms F of the module V onto its g invariant subspace, W, $0 \neq W \neq V$, $H^1_\sigma(g, F) = 0$, σ defining the representation of g in F, F being defined as follows:

$$F[V] = W \qquad F[W] = \{0\}. \tag{1.1}$$

In this paper, we relate the aforesaid cohomology group $H^1_\sigma(g, F)$, to another that is easier to compute, i.e. $H^1_\phi(g, W)$, and possesses a more universal significance; it could be the cohomology of an irreducible ϕ module W, for instance. It turns out under a simple assumption that the vanishing of $H^1_\sigma(g, F)$ is exactly equivalent to the vanishing of $H^1_\phi(g, W)$ if the complementary subspace of V to W, $\bar{W}$ is invariant under the second action of g, i.e. if

$$\phi(X)v_1 = a_X w_1 + b_X w_2 \qquad w_1 \in W, v_1, w_2 \in \bar{W} \tag{1.2}$$

$$\phi(Y)w_2 \in \bar{W} \qquad \text{for all } X, Y \in g \tag{1.3}$$

as long as W is not the trivial representation

$$\phi(X)w = 0 \qquad \text{for all } X \in g,\ w \in W. \tag{1.4}$$

We have no reason to believe that the mode of construction of indecomposable representations that we present is exhaustive; nevertheless, for the indecomposable representations of the spl(2, 1) superalgebra [9], first enumerated by Marcu [6], the method we outline does cover all such representations, so much so that each violation of (1.3) yields a non-zero $H^1_\sigma(g, F)$.

The paper is arranged as follows. In § 2 after giving a brief summary of the cohomology and reducibility of the generalised Lie algebras, we outline our method of constructing indecomposable representations. In § 3 we consider the indecomposable representations of the spl(2, 1) superalgebra as an example, and study the efficacy of our method in this connection.

Our results are stated in the language of ΓGLA [7], which enables them to be accessible to a wider class of algebraic structures.

2. Construction of indecomposable representations

Let g be a Γ graded Lie algebra (ΓGLA) over a commutative field $\mathbb{F}$. Let $\phi : g \to \operatorname{End} V$ define a representation of g in V. We define

$$C^n(g, V) = \{f : g^n \to V : f \text{ is } \mathbb{F} \text{ linear and } f(X_1, \ldots, X_i, X_{i+1}, \ldots, X_n)$$
$$= -\varepsilon(|X_i|, |X_{i+1}|) f(X_1, \ldots, X_{i+1}, X_i, \ldots, X_n) X_i \in g,\ i = 1, \ldots, n\} \tag{2.1}$$

$|X_i| \in \Gamma$, being the degree of $X_i \in g$ on the Abelian group Γ. We further define $\delta^n : C^n(g, v) \to C^{n+1}(g, V)$ by

$$\delta^n f(X_1, \ldots, X_{n+1})$$
$$= \sum_{i=1}^{n+1} (-1)^{i+1} \varepsilon\left(\sum_{k=1}^{i-1} |X_k|, |X_i|\right) \varepsilon(|f|, |X_i|) \phi(X_i) f(X_1, \ldots, \hat{X}_i, \ldots, X_{n+1})$$
$$+ \sum_{i<j=1}^{n+1} (-1)^{i+j} \varepsilon\left(\sum_{k=1}^{i-1} |X_k|, |X_i|\right) \varepsilon\left(\sum_{l=1}^{j-1} |X_l|, |X_j|\right)$$
$$\times \varepsilon(|X_j|, |X_i|) f(\langle X_i, X_j \rangle \ldots \hat{X}_i \ldots \hat{X}_j \ldots X_{n+1}) \tag{2.2}$$

$|f| \in \Gamma$ being the degree of f.

We further define

$$C^0(g, V) = \{f : \mathbb{F} \to V, f \text{ is } \mathbb{F} \text{ linear}\} \tag{2.3}$$

and

$$\delta^0 f(X) = \varepsilon(|f|, |X|) \phi(X) f(1) \qquad \forall X \in g. \tag{2.4}$$

It follows that $\delta^n \circ \delta^{n+1} = 0\ \forall n > 0$ so that we may define

$$
\begin{aligned}
&Z^n_\phi(g, V) = \ker \delta^n \\
&B^n_\phi(g, V) = \operatorname{Im} \delta^{n-1} \\
&H^n_\phi(g, V) = Z^n_\phi(g, V)/B^n_\phi(g, V)
\end{aligned}
\tag{2.5}
$$

as the nth cocycles, nth coboundaries and nth cohomology groups respectively [8].

In earlier investigations, it has been proved that the ϕ module V of g is completely reducible into a g-invariant subspace W of V iff $H^1_\sigma(g, F) = 0$ [5, 10] where F is the set of endomorphisms

$$F[V] = W \qquad F[W] = \{0\} \tag{2.6}$$

and

$$\sigma(X)D = \langle \phi(X), D\rangle \qquad \forall X \in g,\ D \in F \tag{2.7}$$

defines a representation of g in F. This was done by invoking a projection operator B of V onto W which is Γ homogeneous of degree zero. It is easily seen that

$$\theta(X) = \langle \phi(X), B\rangle \in Z^1_\sigma(g, F). \tag{2.8}$$

In what follows, we assume the form (2.8) for the first cocycle $Z^1_\sigma(g, F)$. We now show that $H^1_\sigma(g, F) = 0$ is exactly equivalent to $H^1_\phi(g, W) = 0$ under the assumption that (2.8) defines the form of an element of the first cocyle.

From (2.8), we see that θ is Γ homogeneous of degree zero, as B is. Hence, if $H^1_\sigma(g, F) = 0$,

$$\theta(X) = \sigma(X)D \qquad \forall X \in g,\ D \in F. \tag{2.9}$$

This implies that

$$\langle \phi(X), (B - D)\rangle v_\beta = 0 \qquad \forall v_\beta \in V_\beta \tag{2.10}$$

i.e.

$$(B - D)\phi(X)v_\beta = \phi(X)(B - D)v_\beta. \tag{2.11}$$

Let

$$f(X) = \varepsilon(\beta, |X|)(B - D)\phi(X)v_\beta. \tag{2.12}$$

Then it is easy to see that $f \in Z^1_\phi(g, W)$, seeing that $|f| = \beta$ and

$$\delta f(X, Y) = \varepsilon(|f|, |X|)\phi(X)f(Y) - \varepsilon(|f|, |Y|)\varepsilon(|X|, |Y|)\phi(Y)f(X) - f(\langle X, Y\rangle) = 0$$

using (2.11) and (2.12). But, again using (2.11),

$$f(X) = \varepsilon(\beta, |X|)\phi(X)(B - D)v_\beta. \tag{2.13}$$

Hence if $f \in Z^1_\phi(g, W)$ using (2.13), $\exists w$:

$$f(X) = \delta w(X) = \phi(X)w(1)\varepsilon(|w|, |X|)$$

where $w(1) = (B - D)v_\beta$.

Hence

$$H^1_\sigma(g, F) = 0 \Rightarrow H^1_\phi(g, W) = 0. \tag{2.14}$$

We now see that if W is not the trivial representation (1.3) and if (1.2) holds, $H^1_\phi(g, W)=0 \Rightarrow H^1_\sigma(g, F)=0$.

Let

$$f_1(X)=\varepsilon(\beta,|X|)\phi(x)v_\beta \tag{2.15}$$

$$=\varepsilon(\beta,|X|)[a_X w_{|X|+\beta}+b_X w^1_{|X|+\beta}] \tag{2.15a}$$

$\forall X\in g, |X|, \beta\in\Gamma, v_\beta\in \bar{W}_\beta, w_{|X|+\beta}\in W_{|X|+\beta}, w^1_{|X|+\beta}\in \bar{W}_{|X|+\beta}, a_X, b_X\in \mathbb{F}$, a commutative field of characteristic zero. Then it is easy to see that

$$\delta f_1(X, Y)=0 \qquad f_1\in Z^1_\phi(g, V). \tag{2.16}$$

However, if

$$\phi(Y)w^1_{|X|+\beta}\in W_{|X|+|Y|+\beta} \qquad \forall X, Y\in g, \beta\in\Gamma \tag{2.17}$$

then it may be seen using (2.15) and (2.16) and the fact that W is g invariant that

$$f(X)=\varepsilon(\beta,|X|)a_X w_{|X|+\beta}\in Z^1_\phi(g, W). \tag{2.18}$$

If

$$H^1_\phi(g, W)=0 \qquad f(X)=\varepsilon(\beta,|X|)\phi(X)w_\beta \tag{2.19}$$

so that

$$a_X w_{|X|+\beta}=\phi(X)w'_\beta. \tag{2.19a}$$

Now, if β and θ are defined as in (2.8), then

$$\theta(X)v_\beta=[B\phi(X)-\phi(X)B]v_\beta$$

$$=B(a_X w_{|X|+\beta}+b_X w^1_{|X|+\beta})-\phi(X)Bv_\beta]$$

$$=[a_X w_{|X|+\beta}+B(b_X w^1_{|X|+\beta})-\phi(X)Bv_\beta]. \tag{2.20}$$

Choosing

$$Dv_\beta=B(v_\beta-w'_\beta) \tag{2.20a}$$

we see, using (2.19*a*), that (2.20) reduces to

$$\theta(X)v_\beta=[D\phi(X)-\phi(X)D]v_\beta \tag{2.21}$$

using the fact that for $w^1_{|X|+\beta}Bv_\beta=Dv_\beta$ as

$$\phi(Y)w^1_{|X|+\beta}\in \bar{W}_{|X|+|Y|+\beta} \qquad \forall X, Y\in g$$

hence $w^1_{|X|+|Y|+\beta}=0$ in this case.

(2.21) is equivalent to the statement that $\exists D\in F$

$$\theta(X)=\sigma(X)D \qquad \forall X\in g \tag{2.22}$$

i.e.

$$H^1_\sigma(g, F)=0.$$

Hence, if (2.17) is satisfied, $H^1_\phi(g, W)=0$ is equivalent to $H^1_\sigma(g, F)=0$. We see that the statement $H^1_\sigma(g, F)=0$ is stronger than $H^1_\phi(g, W)=0$, albeit the fact that $H^1_\phi(g, W)$ for irreducible representations W has a more universal significance.

The above analysis breaks down if W is the representation

$$\phi(X)v_\beta=0 \qquad \forall X\in g, v_\beta\in W_\beta, \beta\in\Gamma.$$

If a_X in (2.15*a*) is non-zero, we see from (2.18) that

$$f(X) \in Z^1_\phi(g, W).$$

But in this case, even if (2.17) holds

$$f(X) \neq \phi(X) w_\beta = 0$$

and so the canonical choice (2.20*a*) is not possible.

We close this section with a recipe for constructing finite-dimensional indecomposable representations of a semisimple ΓGLA g. We start with all the finite-dimensional irreducible representations W of the ΓGLA and compute $H^1_\phi(g, W)$ for all such representations (methods for doing so and requiring no further information than the commutation relations of the algebra and the structure of its irreducible representations are detailed in [5, 8]). After this, indecomposable representations are constructed in three stages.

(i) We select those representations W_1 for which $H^1_\phi(g, W_1) \neq 0$. For these, one chooses an irreducible representation W_2 such that the parameters of W_2 interlock with those of W_1 through the action of the generator of g such that W_1 is left invariant. Then one can see from the above analysis that for the representation space $V = W_1 \oplus W_2$ ($\oplus$ signifies that W_1 is left invariant), $H^1_\sigma(g, F) \neq 0$, F and σ being defined in (2.6) and (2.7), respectively, with W_1 replacing W. Hence the representation space V constructed above is indecomposable.

An example is the representation space $V = [\frac{1}{2}]_+ \oplus [1]_+$ of the spl(2, 1) superalgebra which has been considered in [6]. Identifying W_1 with $[\frac{1}{2}]_+$, one sees that $H^1_\phi(g, W_1) \neq 0$ is perfectly consistent with the indecomposability of V as $H^1_\sigma(g, F) \neq 0$.

(ii) We next consider the irreducible representations W_1 for which $H^1_\phi(g, W_1) = 0$. We consider all possible ways of interlocking the representations W_1 and W_2 with each other, so that W_1 is g invariant. Let the action of $X \in g$ on W_1 and W_2 be as follows:

$$\phi(X) w_1 \in W_1 \tag{2.23a}$$

$$\phi(X) w_2 = a_X w_1 + b_X w_2' \qquad w_1 \in W_1,\ w_2, w_2' \in W_2. \tag{2.23b}$$

(iii) Then, when the action of the other generators Y of g on W_2 (for which $H^1_\phi(g, W_2)$ need not be zero), one has only to satisfy the condition

$$\phi(Y) w_2 \notin W_2 \tag{2.24}$$

(i.e. the second action of g does not leave the representation space W_2 invariant) by a judicious choice of the parameters that exist (for spl(2, 1) these are α, β, δ, etc), in order to obtain an indecomposable representation

$$V = W_1 \oplus W_2.$$

If the trivial representation is identified with W_1, one may even drop the restriction given by equation (2.24) in order to obtain an indecomposable representation.

The three steps outlined above constitute, under the above analysis, all possible ways of obtaining $H^1_\sigma(g, F) \neq 0$ (the criterion for indecomposability arrived at in [5]) for the representation $V = W \oplus W'$, namely

(1) by choosing an irreducible representation W: $H^1_\phi(g, W) \neq 0$ and interlocking W with other representations W' such that W is g invariant.

(2) If W is such that $H^1_\phi(g, W) = 0$, by interlocking W with other representations W' such that equation (2.24) is satisfied, W being g invariant.

(3) By choosing W to be the trivial representation and interlocking other irreducible representations W' with W so that W is g invariant.

Hence these constitute ways of obtaining indecomposable representations.

For the spl(2, 1) superalgebra, as will be seen in the next section, these yield all finite-dimensional indecomposable representations.

3. Indecomposable representations of the spl(2, 1) superalgebra

The spl(2, 1) superalgebra is tabulated below for the convenience of the reader. There are four even generators (Q_+, Q_-, Q_3, B) and four odd generators (V_+, V_-, W_+, W_-):

$$[Q_3, Q_\pm] = \pm Q_\pm \qquad [Q_+, Q_-] = 2Q_3 \tag{3.1a}$$

$$[B, Q_\pm] = [B, Q_3] = 0 \tag{3.1b}$$

$$[B, V_\pm] = \tfrac{1}{2} V_\pm \qquad [B, W_\pm] = -\tfrac{1}{2} W_\pm \tag{3.2a}$$

$$[Q_3, V_\pm] = \pm\tfrac{1}{2} V_\pm \qquad [Q_3, W_\pm] = \pm\tfrac{1}{2} W_\pm \tag{3.2b}$$

$$[Q_\pm, V_\pm] = [Q_\pm, W_\pm] = 0 \tag{3.3a}$$

$$[Q_\pm, V_\mp] = V_\pm \qquad [Q_\pm, W_\mp] = W_\pm. \tag{3.3b}$$

$$[V_\pm, V_\pm] = [V_\pm, V_\mp] = [W_\pm, W_\pm] = [W_\pm, W_\mp] = 0 \tag{3.4a}$$

$$[V_\pm, W_\pm] = \pm Q_\pm \qquad [V_\pm, W_\mp] = -Q_3 \pm B. \tag{3.4b}$$

The irreducible representations of the superalgebra are specified by two numbers b and q. q is the maximal isospin contained by the irreducible representation (b, q), and b is the eigenvalue of B for the multiplet

$$Q^2\phi_0 = q(q+1)\phi_0 \tag{3.5a}$$

$$B\phi_0 = b\phi_0. \tag{3.5b}$$

Also, if ϕ_0 is the state with highest weight,

$$Q_+\phi_0 = V_+\phi_0 = W_+\phi_0 = 0. \tag{3.6}$$

The representation space of an irreducible representation of spl(2, 1) is generated by the vectors $Q_-^m\phi_0$, $(Q_-^m V_-)\phi_0$, $(Q_-^m W_-)\phi_0$ and $(Q_-^m V_- W_-)\phi_0$, $m \geq 0$ [9]. The corresponding isospin multiplets are denoted respectively by $|b, q\rangle$, $|b+\frac{1}{2}, q-\frac{1}{2}\rangle$, $|b-\frac{1}{2}, q-\frac{1}{2}\rangle$ and $|b, q-1\rangle$ respectively. A typical irreducible representation (b, q), $b \neq \pm q$, contains all these multiplets except for $q = \frac{1}{2}$, $b \neq \pm\frac{1}{2}$ which does not contain $|b, q-1\rangle$. A non-typical (atypical) irreducible representation is specified by $b = \pm q$, and denoted by $[q]_\pm$. If $b = +q$, the representation contains only the $|b, q\rangle$ and $|b+\frac{1}{2}, q-\frac{1}{2}\rangle$ isospin multiplets, if $b = -q$, the representation contains only the isospin multiplet $|b, +q\rangle$ and $|b-\frac{1}{2}, q-\frac{1}{2}\rangle$.

The indecomposable representations of the spl(2, 1) superalgebra have been studied in detail [6, 10]. These are of two kinds.

(1) Those consisting of two or more typical representations, $(b, q) \oplus (b, q) + \ldots$.

(2) Those consisting of two or more typical representations of the form

(*a*) $[q]_\pm \oplus [q-\frac{1}{2}]_\pm$

(*b*) $[q-\frac{1}{2}]_\pm \oplus [q]_\pm$

(*c*) $[q]_\pm \oplus [q]_\pm$.

It has been seen in [10] that no other form of indecomposable representation can exist for the superalgebra, and also for each of the above $H^1_\sigma(g, F) \neq 0$, and that

$H^1_\phi(g, W)$ is zero for all irreducible finite-dimensional representations W except $[\frac{1}{2}]_\pm$. We now apply the analysis of § 2.

3.1. *Indecomposable representations consisting of two or more typical representations* (b, q)

The matrices of the superalgebra for this representation are given in [6, 10]. The indecomposability of such a representation arises from the fact that the u(1) generator of the Cartan subalgebra is non-diagonal, being in its Jordan canonical form. We note that $H^1_\phi(g, W) = 0$ for all such irreducible representations W.

We begin with the isospin multiplet $|b, q, q_3\rangle_2$ with $X = B$. Then, using (2.15*a*),

$$\phi(B)|b, q, q_3\rangle_2 = b|b, q, q_3\rangle_1 + |b, q, q_3\rangle_2 \tag{3.7}$$

$$a_X = b \qquad b_X = 1. \tag{3.7a}$$

A further application of $\phi(B)$ on $|b, q, q_3\rangle_2$ gives the same result. Hence, by the above analysis, the representation is indecomposable. It also turns out that $H^1_\sigma(g, F) \neq 0$ for the state vector under consideration [10].

The same analysis for $\phi(B)$ may be repeated on each of the other states of (b, q), e.g. $|b+\frac{1}{2}, q-\frac{1}{2}, q_3\rangle$, with the same result. We have thus seen that $H^1_\sigma(g, F) \neq 0$.

We now see that each violation of (2.17) gives rise to a non-vanishing $H^1_\sigma(g, F)$. For instance, considering

$$\begin{aligned}\phi(V_\pm)|b-\tfrac{1}{2}, q-\tfrac{1}{2}, q_3\rangle_2 &= \varepsilon(q \pm q_3 + \tfrac{1}{2})^{1/2}|b, q, q_3 \pm \tfrac{1}{2}\rangle_2 \pm \zeta(q \mp q_3 - \tfrac{1}{2})^{1/2}|b, q-1, q_3 \pm \tfrac{1}{2}\rangle_2 \\ &\quad + \varepsilon'(q \pm q_3 + \tfrac{1}{2})^{1/2}|b, q, q_3 \pm \tfrac{1}{2}\rangle_1.\end{aligned}$$

We have already seen that the action $\phi(B)$ on $|b, q, q_3\rangle_2$ and $|b, q-1, q_3\rangle_2$ involves $|b, q, q_3\rangle_1$ and $|b, q-1, q_3\rangle_1$, so that (2.17) is violated. This violation also holds if we consider

$$\phi(V_\pm)|b, q-1, q_3\rangle_2 \qquad \phi(W_\pm)|b+\tfrac{1}{2}, q-\tfrac{1}{2}, q_3\rangle_2$$

and $\phi(W_\pm|b, q-1, q_3\rangle_2$.

In each of these cases $\exists D \in F$:

$$\theta(V_\pm)|b, q-1, q_3\rangle_2 = \sigma(V_\pm)D|b, q-1, q_3\rangle$$

$$\theta(W_\pm)|b+\tfrac{1}{2}, q-\tfrac{1}{2}, q_3\rangle_2 = \sigma(W_\pm)D|b+\tfrac{1}{2}, q-\tfrac{1}{2}, q_3\rangle_2$$

and

$$\theta(W_\pm)|b, q-1, q_3\rangle_2 = \sigma(W_\pm)D|b, q-1, q_3\rangle_2.$$

Also $H^1_\sigma(g, F) \neq 0$ [10].

We have thus seen for the above representation that whenever (2.17) is violated for the action of a certain generator X on a representation space vector v, $\exists D \in F$:

$$\theta(X)v = \sigma(X)Dv.$$

3.2. Indecomposable representations consisting of two atypical constituents

These representations were introduced in [6] and really consist of three types [10]. The matrices of these representations have been detailed in [6, 10].

(*a*) $[q]_\pm \oplus [q-\frac{1}{2}]_\pm$.

We first consider the representation $[q]_+ \oplus [q-\frac{1}{2}]_+$, the analysis for $[q]_- \oplus [q-\frac{1}{2}]_-$ following on the same lines. We identify W with $[q]_+$ and note that $H^1_\phi(g, W)=0$ $\forall q > \frac{1}{2}$ [10]. In this case

$$\phi(V_\pm)|q-\tfrac{1}{2}, q-\tfrac{1}{2}, q_3\rangle$$
$$= \varepsilon(q \pm q_3 + \tfrac{1}{2})^{1/2}|q, q, q_3 \pm \tfrac{1}{2}\rangle \pm \zeta(q \mp q_3 - \tfrac{1}{2})^{1/2}|q, q-1, q_3 \pm \tfrac{1}{2}\rangle.$$

However,

$$\phi(V_\pm)|q, q-1, q_3 + \tfrac{1}{2}\rangle = \tau(q \pm q_3 + \tfrac{1}{2})^{1/2}|q+\tfrac{1}{2}, q-\tfrac{1}{2}, q_3 \pm 1\rangle$$

so that $V_\pm$, acting on $|q, q-1, q_3 \pm \frac{1}{2}\rangle$ gives a vector belonging to the invariant subspace $[q]_+$ if $\tau \neq 0$, violating (2.17). If $\tau = 0$, as when $q=\frac{1}{2}$, condition (2.17) is satisfied, but $H^1_\phi(g, W) \neq 0$ as $W=[\frac{1}{2}]_+$, hence $H^1_\sigma(g, F) \neq 0$, for if $H^1_\sigma(g, F)$ were zero in this case, $H^1_\phi(g, W)$ would also be zero, by (2.14).

For no other generator acting on any other isospin multiplet of V is (2.17) violated. Hence we see that it is only for the multiplets $|q-\frac{1}{2}, q-\frac{1}{2}, q_3\rangle$ and $|q, q-1, q_3\rangle$ and $X = V_\pm$ that

$$\theta(V_\pm)|q, q-1, q_3\rangle \neq \sigma(V_\pm)D|q, q-1, q_3\rangle$$
$$\theta(V_\pm)|q-\tfrac{1}{2}, q-\tfrac{1}{2}, q_3\rangle \neq \sigma(V_\pm)D|q-\tfrac{1}{2}, q-\tfrac{1}{2}, q_3\rangle$$

resulting in a non-vanishing $H^1_\sigma(g, F)$ [10].

For the representation $[q]_- + [q-\frac{1}{2}]_-$, if $V_\pm$ is replaced by $W_\pm$, τ by ω and $|q-\frac{1}{2}, q-\frac{1}{2}, q_3\rangle$ by $|-q+\frac{1}{2}, q-\frac{1}{2}, q_3\rangle$, respectively, (2.17) is violated and

$$\theta(W_\pm)|q+\tfrac{1}{2}, q-\tfrac{1}{2}, q_3\rangle \neq \sigma(W_\pm)D|-q+\tfrac{1}{2}, q-\tfrac{1}{2}, q_3\rangle$$

unless $\omega = 0$ [10].

If $\omega = 0$, when (2.17) is satisfied, $W=[\frac{1}{2}]_-$, for which $H^1_\phi(g, W)=0$.

(*b*) $[q-\frac{1}{2}]_\pm \oplus [q]_\pm$.

We first consider the representation $[q-\frac{1}{2}]_+ \oplus [q]_+$ and identify W with $[q-\frac{1}{2}]_+$. For the invariant subspace $[q-\frac{1}{2}]_+$, $H^1_\phi(g, W)=0\ \forall q>1$. Considering this representation, we see that

$$\phi(W_\pm)|q+\tfrac{1}{2}, q-\tfrac{1}{2}, q_3\rangle$$
$$= \gamma(q \pm q_3 + \tfrac{1}{2})^{1/2}|q, q, q_3 \pm \tfrac{1}{2}\rangle \pm \delta(q \mp q_3 - \tfrac{1}{2})^{1/2}|q, q-1, q_3 \pm \tfrac{1}{2}\rangle. \tag{3.8}$$

However, for $\beta \neq 0$, $\phi(W_\pm)|q, q, q_3\rangle = \pm\beta(q \mp q_3)^{1/2}|q-\frac{1}{2}, q-\frac{1}{2}, q_3 \pm \frac{1}{2}\rangle$, and hence (2.17) is not satisfied. It has been shown in [10] that $\exists D \in F$

$$\phi(W_\pm)|q+\tfrac{1}{2}, q-\tfrac{1}{2}, q_3\rangle = \sigma(W_\pm)D|q+\tfrac{1}{2}, q-\tfrac{1}{2}, q_3\rangle \tag{3.9a}$$

and

$$\phi(W_\pm)|q, q, q_3\rangle = \sigma(W_\pm)D|q, q, q_3\rangle \tag{3.9b}$$

unless $\beta = \delta = 0$. When $q=\frac{1}{2}$, $\delta = 0$ and (3.8) satisfies (2.17) as $b_X = 0$. However, the invariant subspace W is zero, in which case our analysis breaks down completely, as mentioned earlier, for there is no reason to suppose that $\beta = 0$. When $q = 1$, one further has $H^1_\phi(g, W_1) \neq 0$ for $W_1 = [\frac{1}{2}]_+$, so the representation $[\frac{1}{2}]_+ \oplus [1]_+$ is indecomposable unless $\beta = \delta = 0$.

For the representation $[q-\frac{1}{2}]_- \oplus [q]_-$, $W_\pm$ is to be replaced by $V_\pm$, δ by ζ and β by α and the above remarks may be repeated.

(*c*) $[q]_{1\pm} \oplus [q_2]_{2\pm}$.

We first consider $[q]_{1+} \oplus [q]_{2+}$. Identifying $[q]_{1+}$ with W, we note that $H^1_\phi(g, W) = 0$ $\forall q > \frac{1}{2}$. In this case we see that

$$\phi(V_\pm)|q, q, q_3\rangle_2 = \pm\zeta(q \mp q_3)^{1/2}|q+\tfrac{1}{2}, q+\tfrac{1}{2}, q_3\rangle_1 \pm \pi(q \mp q_3)^{1/2}|q+\tfrac{1}{2}, q-\tfrac{1}{2}, q_3+\tfrac{1}{2}\rangle_2. \quad (3.10a)$$

However, for $\pi \neq 0$

$$\phi(W_\pm)|q+\tfrac{1}{2}, q-\tfrac{1}{2}, q_3+\tfrac{1}{2}\rangle_2 = \theta(q \pm q_3 + 1)^{1/2}|q, q, q_3 \pm 1\rangle_1 + \sigma(q \pm q_3 + 1)^{1/2}|q, q, q_3 \pm 1\rangle_2 \quad (3.10b)$$

in which case, for non-zero θ, (2.17) is violated. It has been shown in [10] that $\exists D \in F$:

$$\theta(V_\pm)|q, q, q_3\rangle_2 = \sigma(V_\pm)D|q, q, q_3\rangle_2 \quad (3.11a)$$

$$\theta(W_\pm)|q+\tfrac{1}{2}, q-\tfrac{1}{2}, q_3\rangle = \sigma(W_\pm)D|q+\tfrac{1}{2}, q-\tfrac{1}{2}, q_3\rangle_2 \quad (3.11b)$$

unless $\pi = \theta = 0$.

When $q = 0$, $\pi = \theta = 0$, so the representation $[q]_{1+} \oplus [q]_{2+}$ is automatically reducible.

For the representation $[q]_1 \oplus [q]_{2-}$, the above analysis may be repeated with the replacement of $V_\pm$ by $W_\pm$, $|q+\frac{1}{2}, q-\frac{1}{2}, q_3\rangle$ by $|-q-\frac{1}{2}, q-\frac{1}{2}, q_3\rangle$, π by ζ and θ by μ, and the above analysis goes through without any changes.

It would be interesting to examine the above method for other irreducible representations of semisimple graded Lie algebras.

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